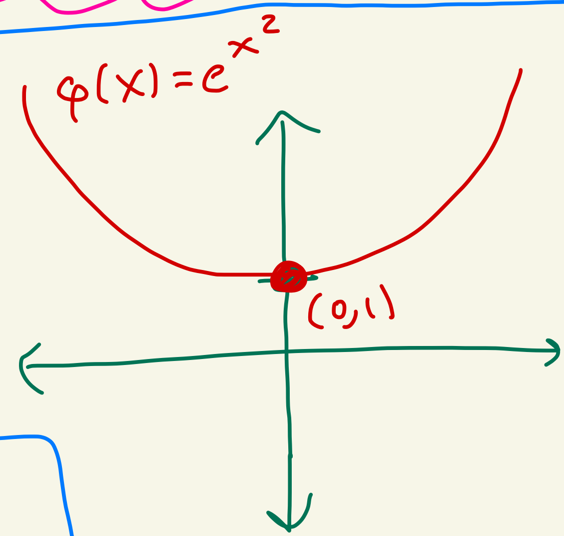
Topic 2 -
First order ODE
Theory

Let us discuss first order ODEs problems of the form

$$\boxed{\begin{aligned} \frac{dy}{dx} &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}}$$

<u>Ex:</u> Consider the initial-value problem

$$(*) \quad \boxed{\begin{aligned} y' &= 2xy \\ y(0) &= 1 \end{aligned}} \longleftarrow \quad y' = \underbrace{2xy}_{f(x,y)=xy}$$



Let $\varphi(x) = e^{x^2}$.

Then, $\varphi'(x) = 2xe^{x^2}$

So, $\varphi'(x) = 2x\varphi(x)$

Thus, φ solves $y' = 2xy$.

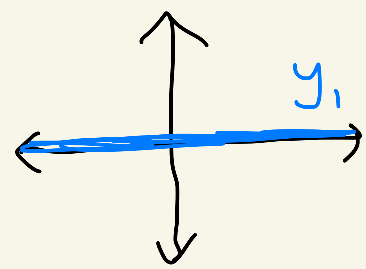
In addition $\varphi(0) = e^{0^2} = e^0 = 1$.

Thus, φ is a solution to $(*)$

It turns out one can show that φ is the only solution to $(*)$. It is unique.

Ex: Consider the initial-value problem

$$\boxed{\begin{aligned} \frac{dy}{dx} &= xy^{1/2} \\ y(0) &= 0 \end{aligned}} \quad (**)$$



Solution 1:

Let $y_1(x) = 0$ for all x.

Then, $y_1' = 0$.

So, $y_1' = 0 = x \cdot 0 = xy^{1/2}$

And, $y_1(0) = 0$.

Thus, y_1 solves $(**)$.

Solution 2:

Let $y_2(x) = \frac{x^4}{16}$.

Then, $y_2' = \frac{1}{4}x^3$

So, $y_2' = \frac{1}{4}x^3$

$$xy_2^{1/2} = x\sqrt{\frac{x^4}{16}} = x \cdot \frac{x^2}{4} = \frac{1}{4}x^3$$

Thus, $y_2' = xy_2^{1/2}$.

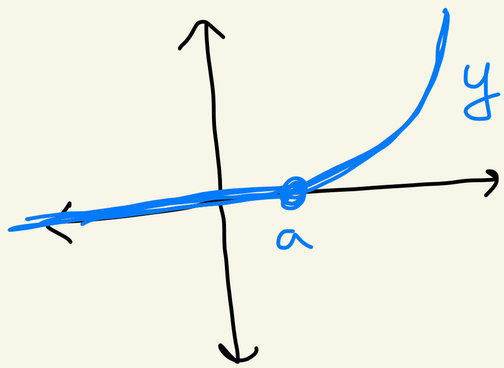
And $y_2(0) = \frac{0^4}{16} = 0$.

Thus, y_2 solves $(**)$ also.

<u>More solutions:</u> Indeed $(**)$ has an infinite # of solutions.

Given $a \geq 0$, let

$$y(x) = \begin{cases} 0, & x < a \\ \frac{(x^2-a^2)^2}{16}, & x \geq a \end{cases}$$



One can show that y solves $(**)$.

<u>Conclusion:</u> $(**)$ has solutions but there is no unique solution.

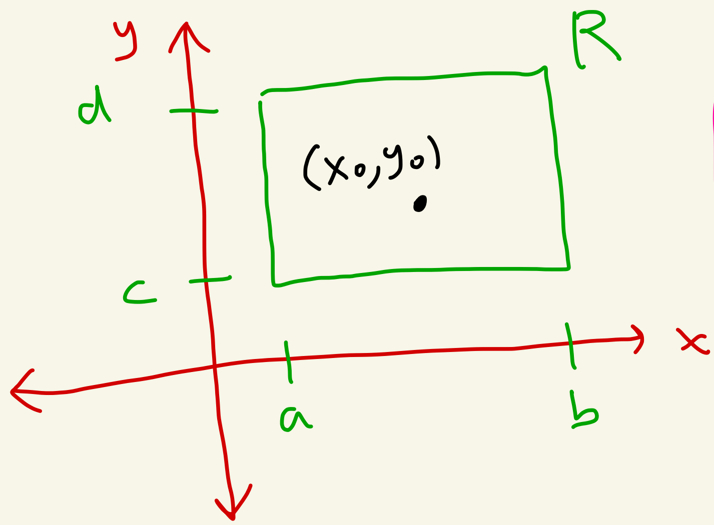
Are there criteria that give a unique solution to the following initial-value problem?

$$\boxed{\begin{aligned} \frac{dy}{dx} &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}}$$

Answer: Yes.

Theorem (due to Picard [1856–1941])

Let R be the rectangular region in the xy-plane defined by $a \le x \le b$ and $c \le y \le d$ that contains the point (x_0, y_0) in its interior.

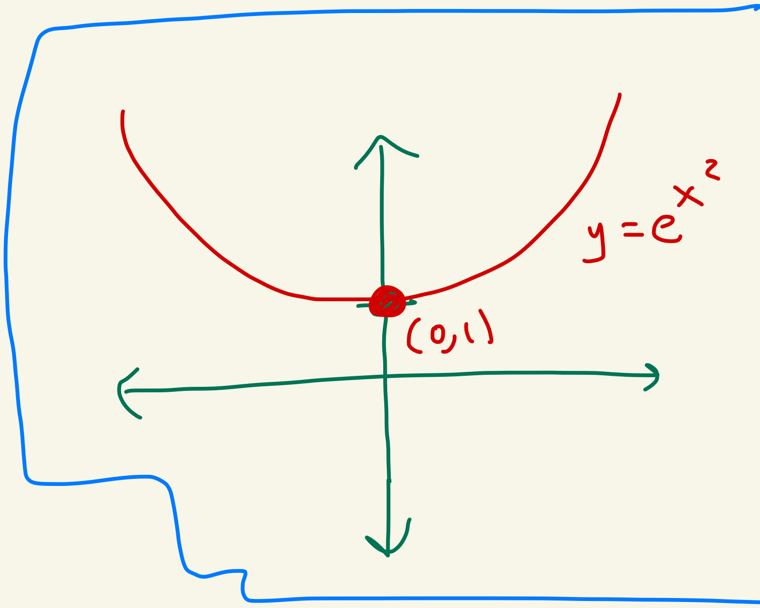


If $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous on R, then there exists an interval I centered at x_0 and a unique function $y(x)$ defined on I that satisfies

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

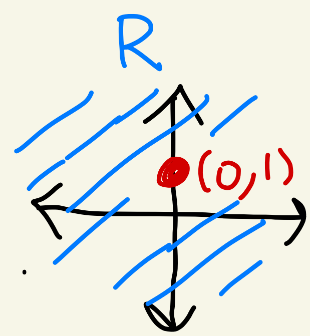
Ex: Consider

$$\boxed{\begin{aligned} y' &= 2xy \\ y(0) &= 1 \end{aligned}}$$



Here $f(x,y) = 2xy$.
We must find a rectangular region R that contains (0,1) where f and $\frac{\partial f}{\partial y}$ are continuous.

$f(x,y) = 2xy$ is continuous everywhere.
$\frac{\partial f}{\partial y} = 2x$ is continuous everywhere.



Thus, R would be the entire xy-plane.
And Picard's thereom says there is a unique solution with $y(0) = 1$.
It is $y(x) = e^{x^2}$ that we saw before.